DTU Library

On the representation of distributions with rational moment generating functions

Bladt, Mogens; Nielsen, Bo Friis

Publication date:
2012

Document Version
Publisher's PDF, also known as Version of record

Citation (APA):
Bladt, M., & Nielsen, B. F. (2012). *On the representation of distributions with rational moment generating functions*. Technical University of Denmark. D T U Compute. Technical Report No. 2012-16

On the representation of distributions with rational moment generating functions

Mogens Bladt[a], Bo Friis Nielsen[b]

[a] Instituto de Investigacion en Matematical Aplicados y en Sistemas, Universidad Nacional Autonoma de Mexico, A.P. 20–726, 01000 Mexico, D.F., Mexico, email: bladt@stats.iimas.unam.mx

[b] Department of Informatics and Mathematical Modelling, Technical University of Denmark, DK–2800 Kgs. Lyngby, Denmark, email: bfn@imm.dtu.dk

Abstract

This paper addresses a question concerning the generality of certain parameterisations of distributions which have a multivariate rational moment generating function. It is shown that the class of bilateral matrix–exponential distributions, as introduced in [2], is strictly larger than the bilateral multivariate matrix–exponential distributions that arise as a generalisation of the the MPH* distributions introduced by Kulkarni [4].

Keywords: multivariate matrix–exponential distribution, bilateral multivariate matrix–exponential distribution, characterisation

1. Introduction

Multivariate distributions on $\mathbb{R}_+^n$ with rational multidimensional moment generating functions, i.e. a fraction between two multidimensional polynomials $p(s)$ and $q(s)$, were introduced and characterised in [3]. If $\boldsymbol{X} = (X_1, ..., X_n)$ is a vector of non–negative random variables and $\boldsymbol{s} = (s_1, ..., s_n)$, then the multidimensional moment generating function of $\boldsymbol{X}$ given by

$$M_X(\boldsymbol{s}) = \mathbb{E}\left(\exp\left(\langle \boldsymbol{s}, \boldsymbol{X} \rangle\right)\right)$$

is a rational function, if and only if $\langle \boldsymbol{s}, \boldsymbol{X} \rangle$ has a univariate matrix–exponential distributions for all non–negative non–zero $\boldsymbol{s}$. Here $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^n$. Such distributions are called multivariate matrix–exponential distributions. Thus for a random vector $\boldsymbol{X}$ with a multivariate

Preprint submitted to Elsevier *September 5, 2012*

matrix–exponential distribution, $\langle \boldsymbol{s}, \boldsymbol{X} \rangle$ has a univariate matrix–exponential distribution which depends on $\boldsymbol{s}$ and thus a representation $(\boldsymbol{\alpha}(\boldsymbol{s}), \boldsymbol{T}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}))$ for some row vector $\boldsymbol{\alpha}(\boldsymbol{s})$, matrix $\boldsymbol{T}(\boldsymbol{s})$ and column vector $\boldsymbol{t}(\boldsymbol{s})$. Thus $\langle \boldsymbol{s}, \boldsymbol{X} \rangle$ is a random variable with density function

$$f(x) = \boldsymbol{\alpha}(\boldsymbol{s}) e^{\boldsymbol{T}(\boldsymbol{s})x} \boldsymbol{t}(\boldsymbol{s}).$$

Sub–classes of the multivariate matrix–exponential distributions have previously been considered by [1] and [4], where the latter construction contains the former. The class defined in [4] is characterised by having a moment generating function that can be expressed as

$$M_{\boldsymbol{X}}(\boldsymbol{s}) = \boldsymbol{\alpha} \left(\boldsymbol{I} + \boldsymbol{T}^{-1} \boldsymbol{\Delta}(\boldsymbol{K}\boldsymbol{s}) \right)^{-1} \boldsymbol{e} = \boldsymbol{\alpha} \left(\boldsymbol{I} - \boldsymbol{U} \boldsymbol{\Delta}(\boldsymbol{K}\boldsymbol{s}) \right)^{-1} \boldsymbol{e}, \tag{1}$$

where $\boldsymbol{\alpha}$ is a p-dimensional row vector, $\boldsymbol{T}$ is some fixed $p \times p$ sub–intensity matrix, $\boldsymbol{K} \geq \boldsymbol{0}$ is a $p \times n$ matrix, $\boldsymbol{\Delta}(\boldsymbol{K}\boldsymbol{s})$ denotes the diagonal matrix with diagonal elements according to the vector $\boldsymbol{K}\boldsymbol{s}$, and $\boldsymbol{e}$ is a p-dimensional column vector of ones. We have $\boldsymbol{U} = (-\boldsymbol{T})^{-1}$. Whenever $\boldsymbol{K}\boldsymbol{s} > 0$ we can express the sub–intensity matrix $\boldsymbol{T}(\boldsymbol{s})$ on the form $\boldsymbol{T}(\boldsymbol{s}) = \boldsymbol{\Delta}^{-1}(\boldsymbol{K}\boldsymbol{s})\boldsymbol{T}$, such that the representation $(\boldsymbol{\alpha}(\boldsymbol{s}), \boldsymbol{T}(\boldsymbol{s}), \boldsymbol{t}(\boldsymbol{s}))$ becomes $(\boldsymbol{\alpha}, \boldsymbol{\Delta}^{-1}(\boldsymbol{K}\boldsymbol{s})\boldsymbol{T}, -\boldsymbol{\Delta}^{-1}(\boldsymbol{K}\boldsymbol{s})\boldsymbol{T}\boldsymbol{e})$. We shall refer to the condition (1) as a MPH* form. In the case of multivariate matrix–exponential distributions the parameters $(\boldsymbol{\alpha}, \boldsymbol{T}, \boldsymbol{K})$ can be general with the requirement that $M_{\boldsymbol{X}}(\boldsymbol{s})$ in Equation (1) expresses a valid moment generating function. The question arising, is whether there exist multivariate matrix–exponential distributions which cannot be written on this particular form. We have so far been unable to successfully address this question in any direction and we shall not provide an answer in this article either. However, we proved in [3] that representations on this form cannot in general be of minimal order, i.e. the order induced by the rational function.

In this paper we shall address a similar question to distributions on $\mathbb{R}^p$ which have rational moment generating functions, i.e. for the so–called bilateral multivariate matrix–exponential distributions. Here the answer is affirmative: there do exist distributions with representations where $\boldsymbol{T}(\boldsymbol{s})$ cannot be written on the MPH* form for any dimension.

Bilateral multivariate matrix–exponential distributions have been considered in [2] and an equivalent characterisation were proved: a vector $\boldsymbol{X} = (X_1, ..., X_n)$ has a bilateral matrix–exponential distribution if and only if for all $\boldsymbol{s} = (s_1, ..., s_n) \neq 0$ the random variables $\langle \boldsymbol{s}, \boldsymbol{X} \rangle$ have a univariate bilateral matrix–exponential distribution. In the following we construct an example of a bivariate and bilateral exponential distribution which cannot be represented on the proposed form.

2. Counter example for bilateral bivariate matrix–exponential distributions

Consider two independent Brownian motions $B_1(t)$ and $B_2(t)$ with zero drift and diffusion coefficients $\sigma_1 > 0$ and $\sigma_2 > 0$ respectively. Hence $B_i(t) \sim \mathrm{N}(0, \sigma_i^2 t)$, $i = 1, 2$. Let T be exponentially distributed with intensity $\lambda > 0$ and define $\boldsymbol{Y} = (B_1(T), B_2(T))$.

Theorem 2.1. *The distribution of $\boldsymbol{Y}$ is a bivariate bilateral matrix–exponential distribution which cannot be written on the BMME* form.*

Proof: The moment generating function $\boldsymbol{Y}$ is given by

$$\begin{aligned} M_{\boldsymbol{Y}}(\boldsymbol{s}) &= \mathbb{E}\left(e^{\langle \boldsymbol{s}, \boldsymbol{Y} \rangle}\right) \\ &= \int_0^\infty \lambda e^{-\lambda x} \mathbb{E}\left(e^{s_1 B_1(t) + s_2 B_2(t)}\right) dt \\ &= \int_0^\infty \lambda e^{-\lambda x} e^{\frac{1}{2}\sigma_1^2 t s_1^2} e^{\frac{1}{2}\sigma_2^2 t s_2^2} dt \\ &= \frac{\lambda}{\lambda - \frac{1}{2}\sigma_1^2 s_1^2 - \frac{1}{2}\sigma_2^2 s_2^2}. \end{aligned}$$

Now assume that $\sigma_1 = \sigma_2 = \sqrt{2}$ and $\lambda = 1$ so that

$$M_{\boldsymbol{Y}}(\boldsymbol{s}) = \frac{1}{1 - s_1^2 - s_2^2}.$$

The distribution of $\boldsymbol{Y}$ is evidently bilateral bivariate matrix–exponential, and assume that it has a representation on the MPH* form. Then there exists matrices $\boldsymbol{T}$ and $\boldsymbol{K}$ and a vector $\boldsymbol{\alpha}$ such that

$$M_{\boldsymbol{Y}}(\boldsymbol{s}) = \boldsymbol{\alpha} \left(\boldsymbol{I} - \boldsymbol{U}\boldsymbol{\Delta}(\boldsymbol{K}\boldsymbol{s})\right)^{-1} \boldsymbol{e} = \boldsymbol{\alpha} \sum_{i=0}^{\infty} \left(\boldsymbol{U}\boldsymbol{\Delta}(\boldsymbol{K}\boldsymbol{s})\right)^i \boldsymbol{e},$$

where $\boldsymbol{U} = (-\boldsymbol{T})^{-1}$. Let $P_i(\boldsymbol{s}) = \boldsymbol{\alpha} \left(\boldsymbol{U}\boldsymbol{\Delta}(\boldsymbol{K}\boldsymbol{s})\right)^i \boldsymbol{e}$ be the i'th term of the sum and let p be the dimension of $\boldsymbol{T}$. The polynomials $P_i(\boldsymbol{s})$ are sums of ith order monomials in s_1, s_2. From the Cayley-Hamilton theorem we can deduce that

$$P_m(\boldsymbol{s}) = \sum_{j=0}^{m-1} a_{m-j}(\boldsymbol{s}) P_j(\boldsymbol{s})$$

where $a_j(\boldsymbol{s})$ are sums of multidimensional monomials of order j, which can be obtained from the characteristic polynomial of $\boldsymbol{U}\boldsymbol{\Delta}(\boldsymbol{K}\boldsymbol{s})$. Consider now the bivariate moment generating function

$$\sum_{i=0}^{\infty} P_i(\boldsymbol{s}) = \frac{1}{1-(s_1^2+s_2^2)} = \sum_{i=0}^{\infty}\left(s_1^2+s_2^2\right)^i .$$

Then

$$P_{2i}(\boldsymbol{s}) = \left(s_1^2+s_2^2\right)^i \quad \text{and} \quad P_{2i-1}(\boldsymbol{s}) = 0 \quad i=0,1,2,...$$

Let us assume that $p=2k$. From the Cayley-Hamilton theorem we have

$$P_{2k}(\boldsymbol{s}) = \left(s_1^2+s_2^2\right)^k = \sum_{j=0}^{k-1} a_{2k-2j}(\boldsymbol{s})\left(s_1^2+s_2^2\right)^j$$

and we can deduce that

$$a_{2k}(\boldsymbol{s}) = \left(s_1^2+s_2^2\right)\left[\left(s_1^2+s_2^2\right)^{k-1} - \sum_{j=1}^{k-1} a_{2k-2j}(\boldsymbol{s})\left(s_1^2+s_2^2\right)^{j-1}\right].$$

But

$$\begin{aligned} a_{2k}(\boldsymbol{s}) &= \mathrm{Det}\left(\boldsymbol{U}\boldsymbol{\Delta}(\boldsymbol{K}\boldsymbol{s})\right) \\ &= \mathrm{Det}\left(\boldsymbol{U}\right)\mathrm{Det}\left(\Delta(\boldsymbol{K}\boldsymbol{s})\right) \\ &= \mathrm{Det}\left(\boldsymbol{U}\right)\prod_{i=1}^{2k}(K_{i1}s_1+K_{i2}s_2), \end{aligned}$$

and it is impossible to get a factor of the form $s_1^2+s_2^2$, contradicting the conjecture that the distribution can be written on the MPH* form with even $p=2k$. With minor modifications the proof also hold for $p=2k-1$ odd. In this case we have

$$P_{2k-1}(\boldsymbol{s}) = 0 = \sum_{j=0}^{k-1} a_{2k-1-2j}(\boldsymbol{s})\left(s_1^2+s_2^2\right)^j$$

leading to

$$a_{2k-1}(\boldsymbol{s}) = -\left(s_1^2+s_2^2\right)\left(\sum_{j=1}^{k-1} a_{2k-1-2j}(\boldsymbol{s})\left(s_1^2+s_2^2\right)^{j-1}\right),$$

and the same argument regarding the determinant applies.

The general case where σ_1 or σ_2 are different from $\sqrt{2}$ and $\lambda \neq 1$ can be obtained as above by normalisation. Q.E.D.

[1] David Assaf, Naftali A. Langberg, Thomas H. Savits, and Moshe Shaked. Multivariate phase-type distributions. *Operations Research*, 32(3):688–702, may-jun 1984.

[2] Mogens Bladt, Luz Judith Rodriguez Esparza, and Bo Friis Nielsen. Bilateral matrix–exponential distributions. In *The Seventh International Conference on Matrix-Analytic Methods in Stochastic Models (MAM7)*, June 2011.

[3] Mogens Bladt and Bo Friis Nielsen. Multivariate matrix–exponential distributions. *Stochastic Models*, 26(1):1–26, 2010.

[4] V. G. Kulkarni. A new class of multivariate phase type distributions. *Operations Research*, 37(1):151–158, jan-feb 1989.